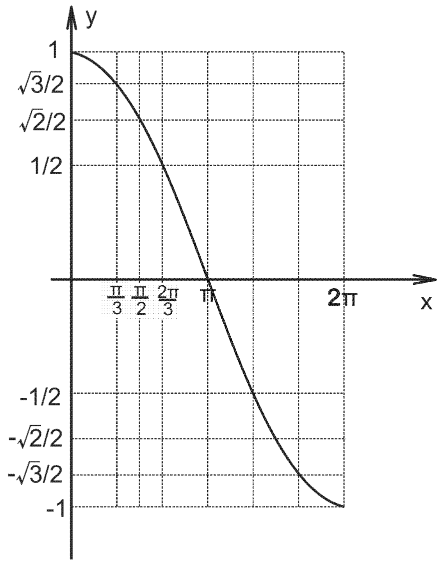
Prvi parcijalni iz Analize III, 26.11.2013.
ispit pisati isključivo hemijskom olovkom, prije rješenja prepisati tekst zadatka



1. Dio grafika funkcije $y = f(x)$ prikazan je na slici lijevo. Datu funkciju pretvoriti u Furijer-ov red samo po cos-inusima. Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=1}^{\infty} \frac{1}{n}$.

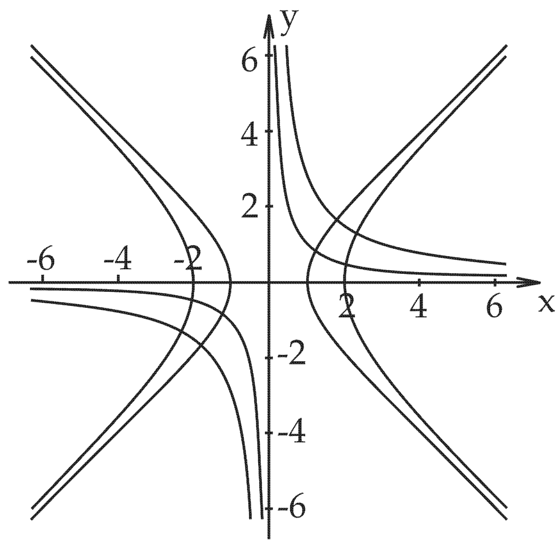
2. (40%)(a) Ako postoji, izračunati limes: $\lim_{(x,y)\to(0,0)} \frac{\sin(x)\cdot\sin(y)}{x^2+y^2}$.

(60%)(b) Izračunati $\lim_{(x,y)\to(0,0)} \frac{y(2^x-1)}{\sin(xy)}$.

3. Odrediti stacionarne tačke funkcije $z(x,y) = 6x^2y - \frac{9}{2}xy^2 - 12xy + 9y^2$.

4. (30%) Dati dvostruki integral $\int_0^2 \mathrm{d}x \int_0^x f(x,y)\mathrm{d}y$ iz pravougaonih transformisati na polarne koordinate.

(70%) Izračunati $I = 4\iint\limits_D xy(x^2+y^2)e^{(xy)^2}\mathrm{d}x\mathrm{d}y$ gdje je $D = \{(x,y) : 1 \leq x^2 - y^2 \leq 4, 1 \leq xy \leq 3, x > 0, y > 0\}$ (za izgled datih krivih vidi sliku desno).



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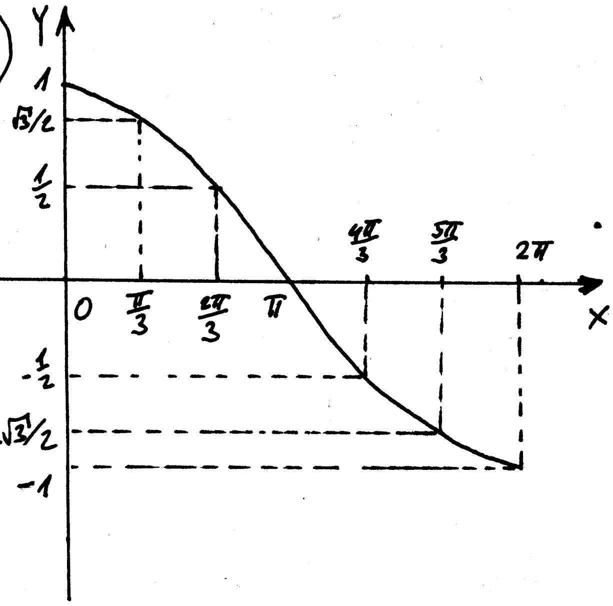


Dio grafika f-je $y=f(x)$ prikazan je na slici lijevo. Datu f-ju pretvoriti u Furijer-ov red samo po cos-inusima. Dobijeni rezultat iskoristiti za sumiranje reda

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Rj: Prepoznajmo prvo koji dio f-je je u pitanju

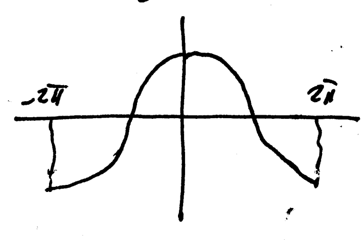
$f(0)=0$, $f(\frac{\pi}{3}) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, $f(\frac{2\pi}{3}) = \cos\frac{2\pi}{6} = \cos\frac{\pi}{3} = \frac{1}{2}$, $f(\pi) = \cos\frac{\pi}{2} = 0$, ...

Nije teško primjetiti da je u pitanju f-ja $y=\cos\frac{x}{2}$.

Jednačina Furijerovog reda je $\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi x}{b-a} + b_n \sin\frac{2n\pi x}{b-a}\right)$

iz čega vidimo da ako želimo da f-ju pretvorimo u Furijerov red samo po cosinusima, trebamo produžiti f-ju tako da je $b_n = 0$.

$$b_n = \frac{2}{b-a}\int_a^b f(x)\, \underbrace{\sin\frac{2n\pi x}{b-a}}_{neparna}\, dx$$



Pravimo parno produženje f-je $F(x) = \begin{cases} \cos\frac{x}{2}, & x\in[0,2\pi) \\ \cos\frac{x}{2}, & x\in(-2\pi,0) \end{cases}$

Primjetimo da je period 4π, $[a,b] = [-2\pi, 2\pi]$

$b-a = 4\pi$, $\frac{2}{b-a} = \frac{1}{2\pi}$, $\frac{2n\pi x}{b-a} = \frac{2n\pi x}{4\pi} = \frac{nx}{2}$

$$a_0 = \frac{2}{b-a}\int_a^b f(x)\,dx = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} \cos\frac{x}{2}\,dx = \left| \begin{matrix} \text{f-ja je} \\ \text{parna} \end{matrix} \right| = \frac{1}{\pi}\int_0^{2\pi} \cos\frac{x}{2}\,dx =$$

$$= \left| \begin{matrix} d\left(\frac{x}{2}\right) = \frac{1}{2}dx \\ dx = 2\,d\left(\frac{x}{2}\right) \end{matrix} \right| = \frac{2}{\pi}\int_0^{2\pi} \cos\frac{x}{2}\, d\left(\frac{x}{2}\right) = \frac{2}{\pi}\sin\frac{x}{2}\Big|_0^{2\pi} = 0$$

$$a_n = \frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} \underbrace{\underbrace{\cos\frac{x}{2}}_{\text{parna}}\,\underbrace{\cos\frac{nx}{2}}_{\text{parna}}}_{\text{parna}}\,dx = \frac{1}{\pi}\int_0^{2\pi}\cos\frac{x}{2}\cos\frac{nx}{2}\,dx$$

$$= \left| \begin{matrix} \cos\left(\frac{x}{2}+\frac{nx}{2}\right) = \cos\frac{x}{2}\cos\frac{nx}{2} - \sin\frac{x}{2}\sin\frac{nx}{2} \\ \cos\left(\frac{x}{2}-\frac{nx}{2}\right) = \cos\frac{x}{2}\cos\frac{nx}{2} + \sin\frac{x}{2}\sin\frac{nx}{2} \\ \hline \cos\frac{x}{2}\cos\frac{nx}{2} = \frac{1}{2}\left(\cos\frac{1+n}{2}x + \cos\frac{1-n}{2}x\right) \end{matrix} \right| = \frac{1}{2\pi}\int_0^{2\pi}\left(\cos\frac{n+1}{2}x + \cos\frac{n-1}{2}x\right)dx$$

Iz ovoga integrala odmah možemo vidjeti da će samo a_1 biti različit od nule, dok su $a_n = 0$ za $n \geq 2$. Provjerimo

$$a_1 = \frac{1}{2\pi}\int_0^{2\pi}(\cos x + \cos 0x)\,dx = \frac{1}{2\pi}(-\sin x)\Big|_0^{2\pi} + \frac{1}{2\pi}x\Big|_0^{2\pi} = 1$$

primijetimo da ovaj dio nije definiran za $n=1$!!!

za $n \geq 2$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi}\left(\cos\frac{n+1}{2}x + \cos\frac{n-1}{2}x\right)dx = \frac{1}{2\pi}\left(-\frac{2}{n+1}\sin\frac{n+1}{2}x\Big|_0^{2\pi} - \frac{2}{n-1}\sin\frac{n+1}{2}x\Big|_0^{2\pi}\right) = 0$$

Kako je f-ja parna to je $b_n = 0$, pa je traženi Furijerov red

$$f(x) \sim \sum_{n=1}^{\infty} a_n\cos\frac{nx}{2} = \cos\frac{x}{2}$$

Od ranije znamo da je red $\sum_{n=1}^{\infty}\frac{1}{n}$ divergentan red.

Dobijeni rezultat ne ~~možemo~~ iskoristiti za sumiranje ovog reda (suma reda je ∞).

(#) Ako postoji, izračunati dati limes

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x)\cdot\sin(y)}{x^2+y^2}$$

Rj:

Izvršimo približavanje tački (0,0) preko x-ose

$$\lim_{(x,y)\to(0,0)} f(x,y) = \left| \begin{array}{l} \text{ako se približavamo} \\ \text{tački preko x-ose} \\ \text{tada je } y=0 \end{array} \right| = \lim_{x\to 0} \frac{\sin(x)\,\sin(0)}{x^2+0^2} = 0$$

Izvršimo približavanje tački (0,0) preko y-ose

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(0,y)\to(0,0)} \frac{\sin(0)\cdot\sin y}{0^2+y^2} = 0$$

Izvršimo približavanje tački (0,0) preko prave $y=x$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,x)\to(0,0)} \frac{\sin(x)\cdot\sin(x)}{x^2+x^2} = \lim_{x\to 0}\left(\frac{1}{2}\cdot\frac{\sin^2 x}{x^2}\right) = \frac{1}{2}\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2$$

$$= \frac{1}{2}$$

Možemo zaključiti da dati limes ne postoji.

(#) Izračunati $\lim\limits_{(x,y)\to(0,0)} \dfrac{y(2^x-1)}{\sin xy}$.

Rj.

Prisjetimo se iz Analize II da je vrijedilo

$$\underline{\lim_{x\to 0} \frac{\sin x}{x} = 1} \quad i \quad \underline{\lim_{x\to 0} \frac{a^x-1}{x} = \ln a}$$

Imamo

$$\lim_{(x,y)\to(0,0)} \frac{y(2^x-1)}{\sin xy} = \lim_{(x,y)\to(0,0)} \frac{\frac{y(2^x-1)}{xy}}{\frac{\sin xy}{xy}} = \lim_{(x,y)\to(0,0)} \frac{\frac{2^x-1}{x}}{\frac{\sin xy}{xy}} =$$

$$= \frac{\lim\limits_{(x,y)\to(0,0)} \frac{2^x-1}{x}}{\lim\limits_{(x,y)\to(0,0)} \frac{\sin xy}{xy}} = \frac{\ln 2}{1} = \ln 2$$

(#) Odrediti stacionarne tačke f-je

$$z = 6x^2y - \frac{9}{2}xy^2 - 12xy + 9y^2$$

Rj.

$$\frac{\partial z}{\partial x} = 12xy - \frac{9}{2}y^2 - 12y$$

$$\frac{\partial z}{\partial y} = 6x^2 - 9xy - 12x + 18y$$

$$12xy - \frac{9}{2}y^2 - 12y = 0 \quad /:3$$

$$6x^2 - 9xy - 12x + 18y = 0 \quad /:3$$

$$4xy - \frac{3}{2}y^2 - 4y = 0 \quad /\cdot 2$$

$$2x^2 - 3xy - 4x + 6y = 0$$

$$8xy - 3y^2 - 8y = 0$$

$$x(2x - 3y) - 2(2x - 3y) = 0$$

$$8xy - 3y^2 - 8y = 0$$

$$(x-2)(2x-3y) = 0$$

$x - 2 = 0$ ili $2x - 3y = 0$

a)

$$x - 2 = 0$$

$$x = 2$$

$$8 \cdot 2 \cdot y - 3y^2 - 8y = 0$$

$$8y - 3y^2 = 0$$

$$y(8 - 3y) = 0$$

$y = 0$ ili $y = \frac{8}{3}$

$M_1(2; 0)$, $M_2(2; \frac{8}{3})$

b)

$$2x - 3y = 0$$

$$x = \frac{3}{2}y \qquad \cancel{8}^{4} \cdot \frac{3}{\cancel{2}_1}y \cdot y - 3y^2 - 8y = 0$$

$$-8y + 9y^2 = 0$$

$$y(9y - 8) = 0$$

$y_1 = 0 \Rightarrow x_1 = 0$

$y_2 = \frac{8}{9} \Rightarrow x_2 = \frac{3}{2} \cdot \frac{8}{9} = \frac{4}{3}$

$M_3(0; 0)$, $M_4(\frac{4}{3}; \frac{8}{9})$

Stacionarne tačke f-je su

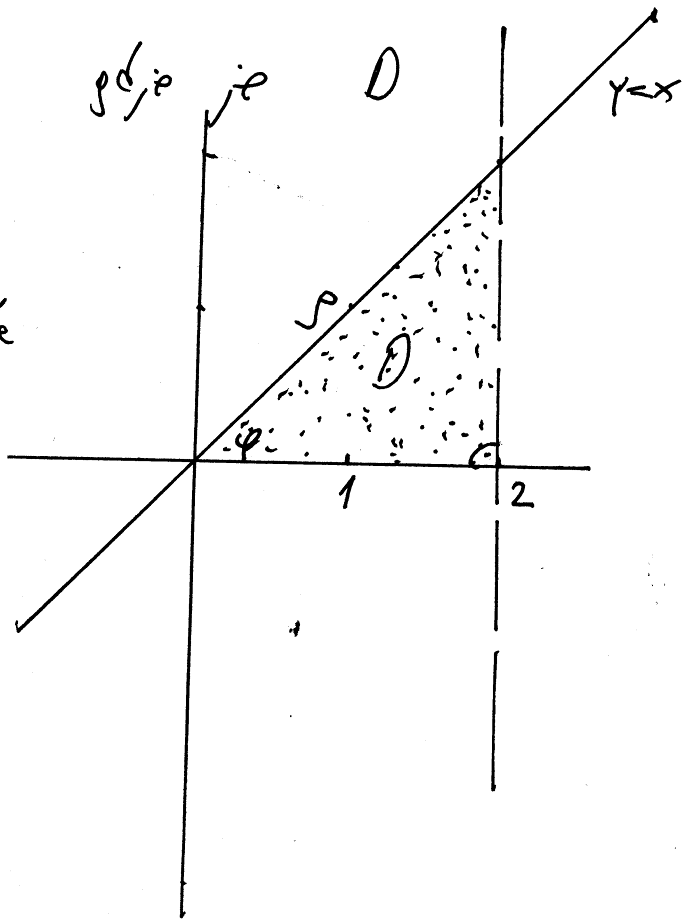
$M_1(2; 0)$, $M_2(2; \frac{8}{3})$, $M_3(0; 0)$, $M_4(\frac{4}{3}; \frac{8}{9})$

Dati dvostruki integral $\int_0^2 dx \int_0^x f(x,y)\,dy$ iz pravougaonih transformisati na polarne koordinate.

Rj.

$$\int_0^2 dx \int_0^x f(x,y)\,dy = \iint_D f(x,y)\,dx\,dy$$

gdje je D



Sa slike:

uvodimo polarne koordinate

$x=\rho\cos\varphi$

$y=\rho\sin\varphi$

$dx\,dy=\rho\,d\rho\,d\varphi$

$$\cos\varphi = \frac{2}{\rho}$$

$$\rho = \frac{2}{\cos\varphi}$$

$$\boxed{J=\frac{D(x,y)}{D(\rho,\varphi)}}$$

$$D \rightsquigarrow D' : \begin{cases} 0 \le \varphi \le \frac{\pi}{4} \\ 0 \le \rho \le \frac{2}{\cos\varphi} \end{cases}$$

Prema tome

$$\int_0^2 dx \int_0^x f(x,y)\,dy = \iint_{D'} \rho f(\rho\cos\varphi, \rho\sin\varphi)\,d\rho\,d\varphi = \int_0^{\frac{\pi}{4}} d\varphi \int_0^{\frac{2}{\cos\varphi}} \rho f(\rho\cos\varphi, \rho\sin\varphi)\,d\rho$$

traženo rješenje

(#) Izračunati $I = 4\iint\limits_D xy(x^2+y^2)e^{(xy)^2}\,dx\,dy$ gdje je

$D = \{(x,y) \mid 1 \le x^2-y^2 \le 4,\ 1 \le xy \le 3,\ x>0,\ y>0\}$.

Rj: Skicirajmo oblast D.

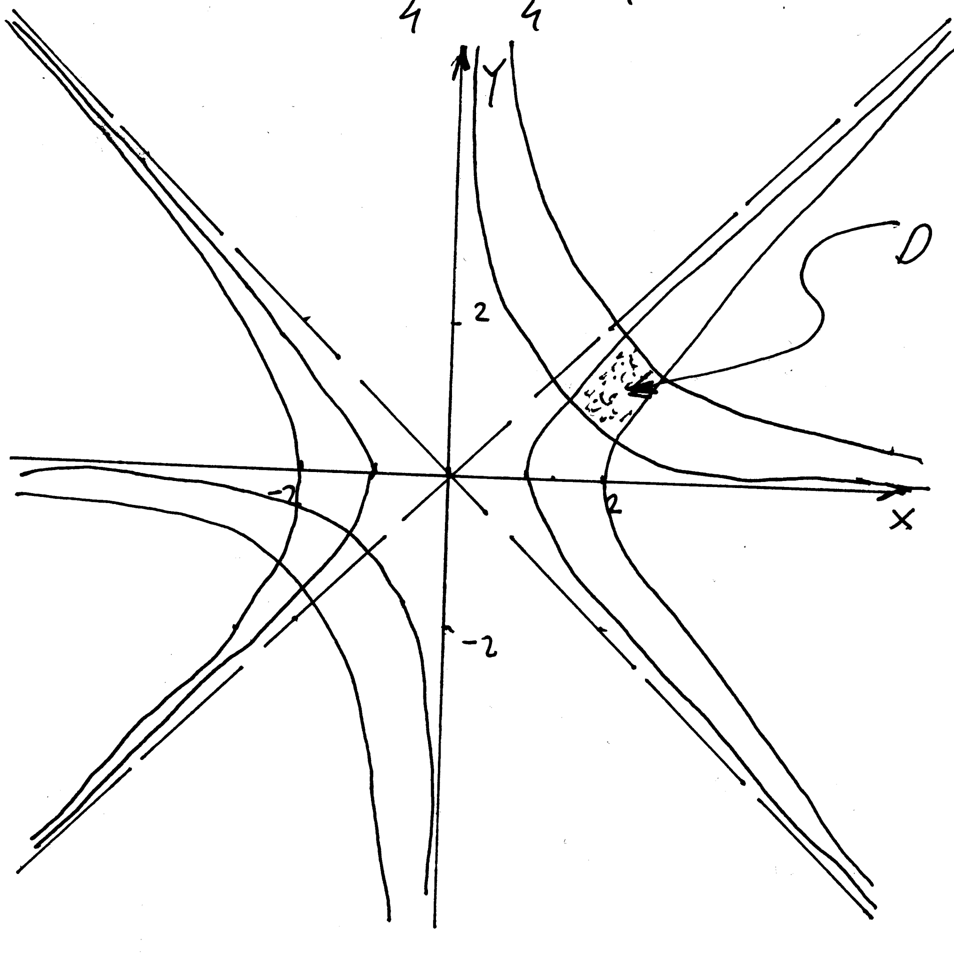
$1 = x^2-y^2$

$x^2-y^2 = 1$

$x^2-y^2=4 \quad /:4$

$\frac{x^2}{4}-\frac{y^2}{4}=1$

Opšti oblik jednačine hiperbole je $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$



$xy=1$

$xy=3$

$y=\frac{1}{x}$

$y=\frac{3}{x}$

$x^2-y^2=1$

$y=\frac{1}{x}$

$x^2-\frac{1}{x^2}=1 \quad /\cdot x^2,\ x\neq 0$

$x^4-x^2-1=0 \quad t=x^2$

$t^2-t-1=0$

Presječne tačke krivih ne igraju nikakvu ulogu u rješavanju zadatka pa ih nećemo tražiti.

Uvedimo smjene $u=x^2-y^2$, $v=xy$, $J^{-1}=\frac{D(u,v)}{D(x,y)}$

$$\frac{D(u,v)}{D(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2x^2+2y^2 = 2(x^2+y^2)$$

$$J^{-1} = 2(x^2+y^2) \Rightarrow J = \frac{1}{2(x^2+y^2)}$$

$$D \xrightarrow{\text{transformare}} D' : \begin{cases} 1 \le u \le 4 \\ 1 \le v \le 3 \end{cases}$$

$$dx\,dy = |J|\,du\,dv = \frac{1}{2} \cdot \frac{1}{x^2+y^2}\,du\,dv$$

$$I = 4 \cdot \frac{1}{2} \iint_{D'} v\,(x^2+y^2) \cdot \frac{1}{x^2+y^2}\,e^{v^2}\,du\,dv$$

$$= 2 \iint_{D'} v e^{v^2}\,du\,dv = \int_1^4 du \int_1^3 2v e^{v^2}\,dv = \left| \, d(v^2) = 2v\,dv \, \right|$$

$$= \int_1^4 du \int_1^3 e^{v^2}\,d(v^2) = u\Big|_1^4 \cdot e^{v^2}\Big|_1^3 = 3(e^9 - e)$$